The Exp-Function Method and *n*-Soliton Solutions

Vangelis Marinakis

Department of Civil Engineering, Technological and Educational Institute of Patras,
1 M. Alexandrou Street, Koukouli, 263 34 Patras, Greece

Reprint requests to V. M.; E-mail: vangelismarinakis@hotmail.com

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We generalize the exp-function method recently proposed by He and Wu [Chaos, Solitons and Fractals **30**, 700 (2006)]. We apply this generalized method to the Korteweg-de Vries equation and derive the known 2-soliton and 3-soliton solutions. We also discuss the efficiency, as well as the drawbacks of the proposed method.

Key words: Exp-Function Method; *n*-Solitons.

1. Introduction

During the last decades, obtaining special solutions of nonlinear integrable and nonintegrable partial differential equations (PDEs) has been a most interesting subject of extensive research. As a result, one can find in the bibliography an enormous amount of books and papers aimed at this direction, where several different kinds of methods are described, such as the inverse scattering method [1], Hirota's method [1], the Painlevé analysis [2 – 4], symmetry reductions [5], the homogenous balance method [6], the tanh method [7], the exp-function method [8], and several "ansatz" methods (see for example [9, 10]).

One disadvantage of many of the methods mentioned above is that they can only lead to wave (or trigonometric) solutions of the form $u = u(x,t) = u(x-ct)$, i. e. solitary waves. This is due to the fact that the starting point of these methods is precisely the assumption that the equation admits a solution of a specific form, which, in every case, embeds in the above general form.

In this paper we generalize the exp-function method, in order to be able to reveal *n*-soliton solutions for any $n \geq 2$. We then apply the proposed method to the famous Korteweg-de Vries (KdV) equation, where the known 2-soliton and 3-soliton solutions are revealed in a simple and straightforward way. We also discuss the advantages, as well as the drawbacks of the proposed generalized method.

2. The Basic Idea

Recently in [8], a new method was proposed for finding special solutions of PDEs, which was called the "exp-function method". The method is based on the assumption that the solutions can be expressed in the form

$$u(x,t) = \sum_{i=0}^{m} a_i \mathrm{e}^{i\xi} \Big/ \sum_{i=0}^{n} b_i \mathrm{e}^{i\xi}, \quad \xi = b(x-ct). \quad (1)$$

By substituting relation (1) into the equation and balancing the highest-order terms we can determine m and n. Consequently, substituting relation (1) into the equation and equating to zero the coefficients of $\mathrm{e}^{i\xi}$, we can determine the coefficients a_i, b_i, b and c (some of them may be arbitrary), or conclude with the fact that the equation does not admit any solution of the above form.

The above method appears to be new, although ideas in this direction can be found in [11 – 13]. It also appears to be reasonable, since the majority of the solutions of various PDEs that appear in the bibliography consist of combinations of exponentials, trigonometric and/or hyperbolic functions, which embed in the form (1).

Since many of the *n*-soliton solutions of various equations are also combinations of exponential functions, it is also reasonable to generalize relation (1) as

$$u(x,t) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} a_{ij} \mathrm{e}^{i\xi_1+j\xi_2} \Big/ \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} b_{ij} \mathrm{e}^{i\xi_1+j\xi_2}, \quad \xi_i = b_i(x-c_i t), \quad i=1,2, \quad (2)$$

or

$$u(x,t) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\sum_{k=0}^{m_3} a_{ijk} \mathrm{e}^{i\xi_1+j\xi_2+k\xi_3} \Big/ \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} b_{ijk} \mathrm{e}^{i\xi_1+j\xi_2+k\xi_3}, \quad \xi_i = b_i(x-c_i t), \quad i=1,2,3. \quad (3)$$

Consequently, substituting relations (2) or (3) into the equation, we can determine (if possible) all coefficients.

Clearly, relations (2) and (3) can now reveal 2-soliton and 3-soliton solutions, respectively. Moreover, we can obviously generalize these relations and reveal in the same way, provided they exist, n-soliton solutions for any fixed n.

3. Application to the KdV Equation

In this section we apply the above procedure to the famous KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{4}$$

and derive the known 2-soliton and 3-soliton solutions. All the computations are carried out by using MATHEMATICA.

3.1. 2-Soliton

We assume that (4) admits a solution of the form

$$u(x,t) = (a_{10}e^{\xi_1} + a_{01}e^{\xi_2} + a_{11}e^{\xi_1+\xi_2} + a_{21}e^{2\xi_1+\xi_2} + a_{12}e^{\xi_1+2\xi_2})(1 + a_1 e^{\xi_1} + a_2 e^{\xi_2} + a_3 e^{\xi_1+\xi_2})^{-2}, \tag{5}$$

where $\xi_i = b_i\,(x - c_i t)$, $i = 1,2$, which embeds in the form (2). Substituting relation (5) in (4) we obtain the relation

$$\frac{\sum_{i=0}^{5}\sum_{j=0}^{5} A_{ij} e^{i\xi_1 + j\xi_2}}{(1 + a_1 e^{\xi_1} + a_2 e^{\xi_2} + a_3 e^{\xi_1+\xi_2})^5} = 0,$$

where $A_{55} = A_{00} = 0$.

Relations $A_{45} = 0$ and $A_{54} = 0$ imply, respectively,

$$c_1 = {b_1}^2, \quad c_2 = {b_2}^2. \tag{6}$$

Consequently, relations $A_{53} = 0$, $A_{35} = 0$ and $A_{44} = 0$ yield, respectively,

$$a_{21} = -2a_1 a_3 {b_2}^2,$$
$$a_{12} = -2a_2 a_3 {b_1}^2,$$
$$a_{11} = -4a_1 a_2 (b_1 - b_2)^2.$$

Then, relation $A_{43} = 0$ implies

$$a_{10} = -\{2a_1 [2a_3 {b_2}^2 (b_1 + b_2)^2 + a_1 a_2 (b_1 - b_2)^2 ({b_1}^2 - 2{b_2}^2)]\}\{a_3 (b_1 + b_2)^2\}^{-1},$$

while relation $A_{34} = 0$ yields

$$a_{01} = -\{2a_2 [2a_3 {b_1}^2 (b_1 + b_2)^2 + a_1 a_2 (b_1 - b_2)^2 ({b_2}^2 - 2{b_1}^2)]\}\{a_3 (b_1 + b_2)^2\}^{-1}.$$

Finally, relation $A_{42} = 0$ implies

$$a_3 = \frac{a_1 a_2 (b_1 - b_2)^2}{(b_1 + b_2)^2}, \tag{7}$$

while the rest of relations $A_{ij} = 0$ are identically satisfied.

Thus, we conclude with the 2-soliton solution (5), where

$$a_{10} = -2a_1 {b_1}^2, \quad a_{01} = -2a_2 {b_2}^2,$$
$$a_{11} = -4a_1 a_2 (b_1 - b_2)^2,$$
$$a_{21} = -\frac{2{a_1}^2 a_2 {b_2}^2 (b_1 - b_2)^2}{(b_1 + b_2)^2},$$
$$a_{12} = -\frac{2a_1 {a_2}^2 {b_1}^2 (b_1 - b_2)^2}{(b_1 + b_2)^2},$$

a_3 is given by (7), c_1 and c_2 are given by (6), and a_1, a_2, b_1, b_2 remain arbitrary.

3.2. 3-Soliton

We now assume that (4) admits a solution of the form

$$u(x,t) = \frac{f_1(\xi_1, \xi_2, \xi_3)}{f_2(\xi_1, \xi_2, \xi_3)}, \tag{8}$$

where $\xi_i = b_i (x - c_i t)$, $i = 1,2,3$, and

$$f_1(\xi_1,\xi_2,\xi_3) = a_{100}e^{\xi_1} + a_{010}e^{\xi_2} + a_{001}e^{\xi_3} + a_{110}e^{\xi_1+\xi_2} + a_{101}e^{\xi_1+\xi_3} + a_{011}e^{\xi_2+\xi_3} + a_{210}e^{2\xi_1+\xi_2} + a_{120}e^{\xi_1+2\xi_2} + a_{201}e^{2\xi_1+\xi_3} + a_{102}e^{\xi_1+2\xi_3} + a_{021}e^{2\xi_2+\xi_3} + a_{012}e^{\xi_2+2\xi_3} + a_{111}e^{\xi_1+\xi_2+\xi_3} + a_{211}e^{2\xi_1+\xi_2+\xi_3} + a_{121}e^{\xi_1+2\xi_2+\xi_3} + a_{112}e^{\xi_1+\xi_2+2\xi_3} + a_{221}e^{2\xi_1+2\xi_2+\xi_3} + a_{212}e^{2\xi_1+\xi_2+2\xi_3} + a_{122}e^{\xi_1+2\xi_2+2\xi_3},$$

$$f_2(\xi_1,\xi_2,\xi_3) = (1 + a_1 e^{\xi_1} + a_2 e^{\xi_2} + a_3 e^{\xi_3} + a_4 e^{\xi_1+\xi_2} + a_5 e^{\xi_1+\xi_3} + a_6 e^{\xi_2+\xi_3} + a_7 e^{\xi_1+\xi_2+\xi_3})^2.$$

Clearly, relation (8) embeds in the form (3).

Substituting relation (8) in (4) we obtain, after similar manipulations,

$$a_{100} = -2a_1 b_1^2, \quad a_{010} = -2a_2 b_2^2, \quad a_{001} = -2a_3 b_3^2, \quad a_{110} = -4a_1 a_2 (b_1 - b_2)^2,$$

$$a_{101} = -4a_1 a_3 (b_1 - b_3)^2, \quad a_{011} = -4a_2 a_3 (b_2 - b_3)^2, \quad a_{210} = -\frac{2a_1^2 a_2 b_2^2 (b_1 - b_2)^2}{(b_1 + b_2)^2},$$

$$a_{120} = -\frac{2a_1 a_2^2 b_1^2 (b_1 - b_2)^2}{(b_1 + b_2)^2}, \quad a_{201} = -\frac{2a_1^2 a_3 b_3^2 (b_1 - b_3)^2}{(b_1 + b_3)^2}, \quad a_{102} = -\frac{2a_1 a_3^2 b_1^2 (b_1 - b_3)^2}{(b_1 + b_3)^2},$$

$$a_{021} = -\frac{2a_2^2 a_3 b_3^2 (b_2 - b_3)^2}{(b_2 + b_3)^2}, \quad a_{012} = -\frac{2a_2 a_3^2 b_2^2 (b_2 - b_3)^2}{(b_2 + b_3)^2},$$

$$a_{111} = -\frac{8a_1 a_2 a_3 \left[b_1^2 \left(b_2^6 + b_3^6\right) - 2b_1^4 \left(b_2^4 + b_3^4\right) + b_1^6 \left(b_2^2 + b_3^2\right) + b_2^2 b_3^2 \left(b_2^2 - b_3^2\right)^2\right]}{(b_1 + b_2)^2 (b_1 + b_3)^2 (b_2 + b_3)^2},$$

$$a_{211} = -\frac{4a_1^2 a_2 a_3 (b_1 - b_2)^2 (b_1 - b_3)^2 (b_2 - b_3)^2}{(b_1 + b_2)^2 (b_1 + b_3)^2}, \; a_{121} = -\frac{4a_1 a_2^2 a_3 (b_1 - b_2)^2 (b_1 - b_3)^2 (b_2 - b_3)^2}{(b_1 + b_2)^2 (b_2 + b_3)^2},$$

$$a_{112} = -\frac{4a_1 a_2 a_3^2 (b_1 - b_2)^2 (b_1 - b_3)^2 (b_2 - b_3)^2}{(b_1 + b_3)^2 (b_2 + b_3)^2}, \; a_{221} = -\frac{2a_1^2 a_2^2 a_3 b_3^2 (b_1 - b_2)^4 (b_1 - b_3)^2 (b_2 - b_3)^2}{(b_1 + b_2)^4 (b_1 + b_3)^2 (b_2 + b_3)^2},$$

$$a_{212} = -\frac{2a_1^2 a_2 a_3^2 b_2^2 (b_1 - b_2)^2 (b_1 - b_3)^4 (b_2 - b_3)^2}{(b_1 + b_2)^2 (b_1 + b_3)^4 (b_2 + b_3)^2}, \; a_{122} = -\frac{2a_1 a_2^2 a_3^2 b_1^2 (b_1 - b_2)^2 (b_1 - b_3)^2 (b_2 - b_3)^4}{(b_1 + b_2)^2 (b_1 + b_3)^2 (b_2 + b_3)^4},$$

$$a_4 = \frac{a_1 a_2 (b_1 - b_2)^2}{(b_1 + b_2)^2}, \quad a_5 = \frac{a_1 a_3 (b_1 - b_3)^2}{(b_1 + b_3)^2}, \quad a_6 = \frac{a_2 a_3 (b_2 - b_3)^2}{(b_2 + b_3)^2},$$

$$a_7 = \frac{a_1 a_2 a_3 (b_1 - b_2)^2 (b_1 - b_3)^2 (b_2 - b_3)^2}{(b_1 + b_2)^2 (b_1 + b_3)^2 (b_2 + b_3)^2},$$

and $c_i = b_i^2$, $i = 1,2,3$, while a_1, a_2, a_3, and b_1, b_2, b_3 remain arbitrary.

4. Conclusions and Discussion

We proposed a generalization of the exp-function method. Consequently, we applied this generalized method to the famous KdV equation and revealed the known 2-soliton and 3-soliton solutions.

The method consists only of algebraic manipulations, which can be carried out by using any computer algebra program, such as MATHEMATICA or MATLAB. Moreover, since the existence of *n*-soliton solutions is a general feature of integrable PDEs, the proposed method could be used not simply to derive new solutions, but actually to reveal new integrable cases.

One drawback of the proposed method is that the balancing of the highest-order terms can actually lead to infinite many cases that have to be treated separately. We should mention though that, regarding the derivation of one-soliton solutions, this problem can be partially overcome, if auxiliary ordinary differential equations are used [14, 15]. Even for fixed m_i and n_i [see relations (2) and (3)] the method becomes rather complicated, depending on these values and the equation itself, since equating to zero the coefficients of the exponentials may imply a highly nonlinear system.

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